Multifractality of the Feigenbaum Attractor and Fractional Derivatives

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It is shown that fractional derivatives of the (integrated) invariant measure of the Feigenbaum map at the onset of chaos have power-law tails in their cumulative distributions, whose exponents can be related to the spectrum of singularities $f(\alpha)$. This is a new way of characterizing multifractality in dynamical systems, so far applied only to multifractal random functions [Frisch and Matsumoto, *J. Stat. Phys.* **108**:1181, 2002]. The relation between the thermodynamic approach [Vul, Sinai and Khanin, *Russian Math. Surveys* **39**:1, 1984] and that based on singularities of the invariant measures is also examined. The theory for fractional derivatives is developed from a heuristic point view and tested by very accurate simulations.

1. INTRODUCTION

Recently a new method for analyzing multifractal functions $u(x)$ was introduced.[(1)] It exploits the fact that the fractional derivative of order a (denoted here by D^a) of $u(x)$ has, for a suitable range of a, a power-law tail in its cumulative probability

$$\mathrm{Prob}\{|D^a u| > \xi\} \propto \xi^{-p_\star}, \quad \xi \to \infty. \tag{1}$$

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The exponent $p_\star(a)$ is the unique solution of the equation

$$\zeta_{p_\star} = p_\star(a)a, \tag{2}$$

where ζ_p is the scaling exponent associated to the behavior at small separations l of the structure function of order p, i.e. $\langle |u(x+l) - u(x)|^p \rangle \sim |l|^{\zeta_p}$. It was also shown that the actual observability of the power-law tail when multifractality is restricted to a finite range of scales is controlled by how much ζ_p departs from linear dependence on p. The larger this departure the easier it is to observe multifractality.

So far the theory of such power-law tails has been developed only for synthetic random functions, in particular the random multiplicative process[2] for which Kesten-type maps[3] and large deviations theory can be used.

It is our purpose here to test the fractional derivative method for invariant measures of dissipative dynamical systems, in particular for the Feigenbaum invariant measure which appears at the accumulation point of the period doubling cascade where the orbit has period 2^∞,[4, 5] Its multifractality was proven rigorously in ref. 6 using a thermodynamic formalism. For the Feigenbaum measure all scaling exponents can be determined with arbitrary accuracy.

There is an important difference in the way one processes functions and invariant measures to determine their multifractal properties and in particular the spectrum of singularities, usually denoted $D(h)$ for functions[7] and $f(\alpha)$ for measures.[8, 9] For a function $u(x)$ one uses the moments or the PDFs of the increments $u(x+l) - u(x)$ to determine the scaling exponents, whereas for an invariant measure μ_0 one works with integrals over intervals or boxes of different sizes. In the one-dimensional case the two approaches become equivalent by introducing the cumulative distribution function

$$u(x) \equiv \int_{-\infty}^{x} \mu_0(dx). \tag{3}$$

Hence we shall apply the fractional derivative method to the integral of the invariant measure.

The organization of the paper is the following. Section 2 is devoted to the thermodynamic formalism for the Feigenbaum attractor. In Section 2.1, we recall the method used in ref. 6. In Section 2.2 we show how this formalism, based on the study of the geometrical properties of the attractor, is actually connected to the standard multifractal formalism which focusses on the statistical properties of the invariant measure.[8,9]

This connection was established by M. Feigenbaum.[10,11] However for convenience of readers we present relevant results here. Then, in Section 2.3 we calculate numerically the free energy and accordingly the scaling exponents ζ_p for the integral of the invariant measure; this is done by a very accurate transfer-matrix-based method. Fractional derivatives are discussed in Section 3. In Section 3.1 we briefly recall the phenomenology of power-law tails in the distribution of fractional derivatives and the limits on observability. The fractional derivative analysis of the Feigenbaum measure is presented in Section 3.2. Concluding remarks are made in Section 4.

2. THERMODYNAMIC FORMALISM FOR THE FEIGENBAUM ATTRACTOR

2.1. Thermodynamic formalism

In this section we give a brief description of the thermodynamic formalism for the invariant measure of the Feigenbaum map (see ref. 6 for the mathematical details) and show how one can use it in order to study the multifractal properties of the Hölder exponents.

By Feigenbaum attractor we understand the attractor of the one-dimensional mapping $g:[0, 1] \to [0, 1]$, where $g(x)$ is the solution of the Feigenbaum–Cvitanović doubling equation:

$$g(x) = -\frac{1}{\gamma} g(g(\gamma x)), \quad g(0) = 1, \quad \gamma = \frac{1}{\alpha} = -g(1). \tag{4}$$

Eq. (4) is known to have the unique solution in the class of smooth unimodal maps (that is, maps having one critical point) with a non-degenerate maximum. This solution is called the Feigenbaum map. It is holomorphic in some complex neighborhood of $[-1, 1]$ and the first few terms in the power series expansion are[12]

$$g(x) = 1 - 1.5276\ldots x^2 + 0.1048\ldots x^4 + 0.0267\ldots x^6 - 0.003527\ldots x^8 + \ldots. \tag{5}$$

The value of the universal constant γ which is the inverse of the Feigenbaum scaling constant α is approximately equal to 0.3995. An attractor A for the map g can be constructed in the following way. For each $n \geqslant 1$

define a collection of intervals of n-th level:

$$\Delta_0^{(n)} = [-\gamma^n, \gamma^n],$$
$$\Delta_i^{(n)} = g^{(i)}(\Delta_0^{(n)}) \equiv \underbrace{g \circ g \circ \cdots \circ g}_{i}\left(\Delta_0^{(n)}\right) \quad (1 \leqslant i \leqslant 2^n - 1). \qquad (6)$$

The following properties of the intervals $\Delta_i^{(n)}$ are easy consequences of the doubling Eq. (4): (a) Intervals $\Delta_i^{(n)}, 0 \leqslant i \leqslant 2^n - 1$ are pairwise disjoint; (b) $g^{(2^n)}\Delta_0^{(n)} \subset \Delta_0^{(n)}$; (c) Each interval of n-th level $\Delta_i^{(n)}$ contains exactly two intervals of $(n + 1)$-th level; $\Delta_i^{(n+1)}$ and $\Delta_{i+2^n}^{(n+1)}$; (d) $2\gamma^{2n} \leqslant |\Delta_i^{(n)}| \leqslant 2\gamma^n$, where $|\cdot|$ denotes the length of the interval. The first three levels of the intervals are shown in Fig. 1.

The properties above imply that it is natural to use a dyadic representation for the intervals $\Delta_i^{(n)}$. Let $i = \sum_{j=0}^{n-1} \epsilon_j 2^i$, where $\epsilon_j = 0, 1$. Then we can use a sequence $(\epsilon_0, \ldots, \epsilon_{n-1})$ as a symbolic coding for intervals $\Delta_i^{(n)}$: $\Delta_i^{(n)} = \Delta_{\epsilon_0, \ldots, \epsilon_{n-1}}^{(n)}$. Now we can define the Feigenbaum attractor

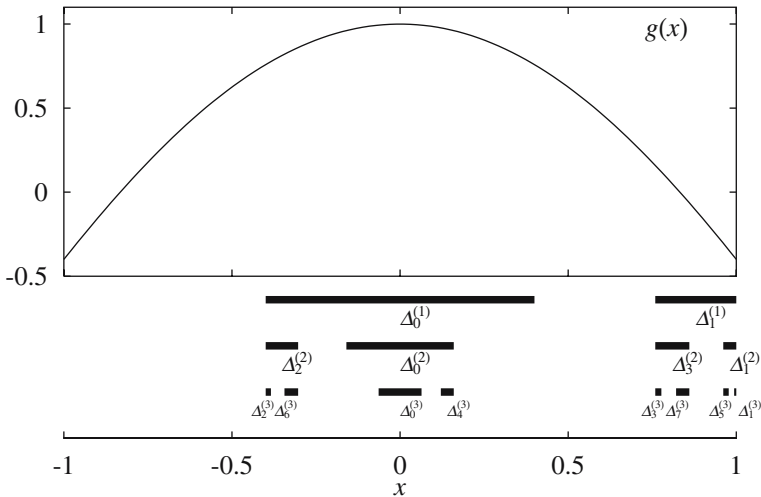$$A = \bigcap_{n \geqslant 1} \bigcup_{i=0}^{2^n - 1} \Delta_i^{(n)}. \qquad (7)$$

Fig. 1.   The Feigenbaum–Cvitanović map $g(x)$ and the first three levels of the partitions $\Delta_j^{(n)}$. For $g(x)$ we used the expansion (5), introduced in ref. 12 up to x^{32}.

The set A is isomorphic to the set of all infinite dyadic sequences $(\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \ldots)$. Such sequences can be considered as a symbolic coordinate system on A. In this new coordinate system the map g acts as the dyadic addition of the sequence $(1, 0, \ldots, 0, \ldots)$. Notice that topologically A is a Cantor set. It is easy to see that A is indeed an attractor for all but countably many initial points $x \in [-1, 1]$: $\operatorname{dist}(g^{(m)}(x), A) \to 0$ as $m \to \infty$. The exceptional set of initial points consists of all unstable periodic orbits and their preimages.

As we have seen above, all intervals $\Delta_i^{(n)}$ have exponentially small lengths but the exponent varies from γ^n to γ^{2n}. Notice that exponents $\ln|\Delta_i^{(n)}|$ give all possible scalings of the fractal set A. The basic ingredient which is needed for the multifractal analysis is the control over the spectrum of possible scalings corresponding to these exponents. Such control can be achieved with the help of the thermodynamic formalism. The thermodynamic formalism which was constructed in ref. 6 is based on the Gibbsian description for the lengths of the intervals $\Delta_i^{(n)}$. It is shown in ref. 6 that there exists a function $U(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(n)}, \ldots)$ (thermodynamic potential) which is defined on all infinite dyadic sequences such that:

I. There exists a constant $C > 0$ for which

$$\frac{1}{C} \leqslant -U(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(n)}, \ldots) \leqslant C. \tag{8}$$

II. For any two dyadic sequences

$$(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(n)}, \bar{\epsilon}^{(n+1)}, \bar{\epsilon}^{(n+2)}, \ldots), (\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(n)}, \tilde{\epsilon}^{(n+1)}, \tilde{\epsilon}^{(n+2)}, \ldots)$$

which coincide on the first n positions

$$\begin{aligned} |U\,(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(n)}, \bar{\epsilon}^{(n+1)}, \bar{\epsilon}^{(n+2)}, \ldots) \\ -U(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(n)}, \tilde{\epsilon}^{(n+1)}, \tilde{\epsilon}^{(n+2)}, \ldots)| \leqslant C(2\gamma)^n. \end{aligned} \tag{9}$$

III. For any $\Delta_{\epsilon_0, \ldots, \epsilon_{n-1}}^{(n)}$ with $\epsilon_0 = 1$

$$\begin{aligned} \exp[-C(2\gamma)^{\frac{n}{3}}] \leqslant \frac{|\Delta_{\epsilon_0, \ldots, \epsilon_{n-1}}^{(n)}|}{|\Delta_{\epsilon_0, \ldots, \epsilon_{n-2}}^{(n-1)}|} \exp(-U(\epsilon_{n-1}, \ldots, \epsilon_1, 1, 0, \ldots, 0, \ldots)) \\ \leqslant \exp[C(2\gamma)^{\frac{n}{3}}]. \end{aligned} \tag{10}$$

It immediately follows from (10) that for $C_1 = \exp\left[\dfrac{C}{1-(2\gamma)^{\frac{1}{3}}}\right]$

$$\frac{1}{C_1} \leqslant \frac{\left|\Delta_{1,\epsilon_1,\ldots,\epsilon_{n-1}}^{(n)}\right|}{\exp\left[\sum_{s=0}^{n-1} U(\epsilon_s, \epsilon_{s-1}, \ldots, \epsilon_1, 1, 0, \ldots, 0, \ldots)\right]} \leqslant C_1. \tag{11}$$

The condition $\epsilon_0 = 1$ corresponding to intervals $\Delta_i^{(n)}$ with odd i's plays only a technical role and it is not essential for our further analysis since the odd intervals contain information about the lengths of the even ones. Indeed, it is very easy to see that for every odd i the intervals $\Delta_i^{(n)}$ and $\Delta_{i+1}^{(n)}$ have lengths of the same order.

We next introduce a parameter β (inverse temperature) and define the partition function

$$Z_n(\beta) = \sum_{\epsilon_{n-1},\ldots,\epsilon_0=0,1} \exp\left[\beta \sum_{s=0}^{n-1} U(\epsilon_s, \epsilon_{s-1}, \ldots, \epsilon_1, 1, 0, \ldots, 0, \ldots)\right] \tag{12}$$

and the free energy

$$F_n(\beta) = \frac{\ln Z_n(\beta)}{n}, \quad F(\beta) = \lim_{n\to\infty} F_n(\beta). \tag{13}$$

It immediately follows from (12) and (13) that

$$\sum_{i=0}^{2^n-1} |\Delta_i^{(n)}|^\beta \sim \exp[nF(\beta)]. \tag{14}$$

In the thermodynamic limit $n \to \infty$ the probability distributions

$$\nu_\beta^{(n)}(\epsilon_{n-1}, \epsilon_{n-2}, \ldots, \epsilon_1)$$
$$= \frac{1}{Z_n(\beta)} \exp\left[\beta \sum_{s=0}^{n-1} U(\epsilon_s, \epsilon_{s-1}, \ldots, \epsilon_1, 1, 0, \ldots, 0, \ldots)\right] \tag{15}$$

tend to a limiting distribution ν_β which can be considered as a Gibbs measure with the potential U, inverse temperature β and the boundary condition $\epsilon_0 = 1$, $\epsilon_{-i} = 0$, $i \in \mathbb{N}$. This Gibbs distribution generates the probability measure on $A_+ = A \cap \Delta_1^{(1)}$ which is the part of the whole attractor

A corresponding to intervals $\Delta_i^{(n)}$ with odd numbers i. We shall denote this Gibbs measure on A_+ by μ_β. Notice that $\beta = 0$ corresponds to a unique invariant measure and $\beta = 1$ gives a conditional distribution corresponding to Lebesgue measure on $[-1, 1]$. The free energy $F(\beta)$ contains all information about the multifractal properties of the Feigenbaum attractor. Notice that the thermodynamic formalism leads to one-dimensional statistical mechanics with exponential decay of interactions and hence without phase transitions. This implies that $F(\beta)$ is a smooth function. In fact it is holomorphic in some complex neighborhood of the real axis. Denote

$$H(\epsilon_{n-1}, \ldots, \epsilon_1) = \sum_{s=0}^{n-1} U(\epsilon_s, \epsilon_{s-1}, \ldots, \epsilon_1, 1, 0, \ldots). \tag{16}$$

Using relations

$$F_n'(\beta) = \frac{1}{Z_n(\beta)} \sum_{\epsilon_{n-1},\ldots,\epsilon_0=0,1} \frac{1}{n} H(\epsilon_{n-1}, \ldots, \epsilon_1) \exp[\beta H(\epsilon_{n-1}, \ldots, \epsilon_1)]$$

$$= \left\langle \frac{\ln |\Delta_{1,\epsilon_1,\ldots,\epsilon_{n-1}}^{(n)}|}{n} \right\rangle_{\nu_\beta^{(n)}} + O\left(\frac{1}{n}\right) \tag{17}$$

and

$$F_n''(\beta) = \frac{1}{Z_n(\beta)} \sum_{\epsilon_{n-1},\ldots,\epsilon_0=0,1} \left(\frac{1}{n} H(\epsilon_{n-1}, \ldots, \epsilon_1)\right)^2 \exp[\beta H(\epsilon_{n-1}, \ldots, \epsilon_1)]$$

$$- \left(\frac{1}{Z_n(\beta)} \sum_{\epsilon_{n-1},\ldots,\epsilon_0=0,1} \frac{1}{n} H(\epsilon_{n-1}, \ldots, \epsilon_1) \exp[\beta H(\epsilon_{n-1}, \ldots, \epsilon_1)]\right)^2$$

$$= \left\langle \left(\frac{\ln |\Delta_{1,\epsilon_1,\ldots,\epsilon_{n-1}}^{(n)}|}{n}\right)^2 \right\rangle_{\nu_\beta^{(n)}} - \left(\left\langle \frac{\ln |\Delta_{1,\epsilon_1,\ldots,\epsilon_{n-1}}^{(n)}|}{n} \right\rangle_{\nu_\beta^{(n)}}\right)^2 + O\left(\frac{1}{n}\right), \tag{18}$$

we conclude that $F(\beta)$ is a monotone decreasing convex function. We shall also use the spectral representation for the free energy. Consider the

transfer-matrix operator $\mathcal{L}(\beta)$:

$$\mathcal{L}(\beta)h(\epsilon^{(1)}, \epsilon^{(2)}, \dots, \epsilon^{(n)}, \dots) = \sum_{\epsilon^{(0)}=0,1} \exp[\beta U(\epsilon^{(0)}, \epsilon^{(1)}, \epsilon^{(2)}, \dots, \epsilon^{(n)}, \dots)]$$
$$\times h(\epsilon^{(0)}, \epsilon^{(1)}, \epsilon^{(2)}, \dots, \epsilon^{(n)}, \dots). \quad (19)$$

Since $\mathcal{L}(\beta)$ is a positive linear operator, its largest eigenvalue $\lambda(\beta)$ is strictly positive and simple. It is easy to see that

$$F(\beta) = \ln \lambda(\beta). \quad (20)$$

For an arbitrary point $x \in A_+$, denote by $\Delta^{(n)}(x)$ the interval of the n-th level which contains x. It follows from (17) that for points x which are typical with respect to μ_β (that is corresponding to a set of full μ_β-measure)

$$|\Delta^{(n)}(x)| \sim \exp[F'(\beta)n]. \quad (21)$$

More precisely, for μ_β-almost all $x \in A$

$$\lim_{n\to\infty} \frac{\ln |\Delta^{(n)}(x)|}{n} = F'(\beta). \quad (22)$$

We next find the total number $N_n(\beta)$ of the intervals of n-th level whose length is of the order $|\Delta^{(n)}(\beta)| = \exp[F'(\beta)n]$. We have

$$N_n(\beta)|\Delta^{(n)}(\beta)|^\beta \sim N_n(\beta)\exp[\beta F'(\beta)n] \sim Z_n(\beta) \sim \exp[F(\beta)n] \quad (23)$$

which gives

$$N_n(\beta) \sim \exp[(F(\beta) - \beta F'(\beta))n]. \quad (24)$$

Using (24) we can find the Hausdorff dimension $d_H(\beta)$ of the set of points $x \in A_+$ which are typical with respect to the measure μ_β. Since

$$N_n(\beta)|\Delta^{(n)}(\beta)|^{d_H(\beta)} \sim \exp[(F(\beta) - \beta F'(\beta) + d_H(\beta)F'(\beta))n] \quad (25)$$

we conclude that

$$F(\beta) - \beta F'(\beta) + d_H(\beta)F'(\beta) = 0 \quad (26)$$

which immediately implies

$$d_H(\beta) = \frac{\beta F'(\beta) - F(\beta)}{F'(\beta)} = \beta - \frac{F(\beta)}{F'(\beta)} . \tag{27}$$

The Hausdorff dimension $d_H(A)$ of the whole attractor A is equal to the maximum of $d_H(\beta)$ over all $\beta \in \mathbb{R}$. Let β_0 be the unique solution of the equation $F(\beta) = 0$. It is easy to see that $d_H(A) = d_H(\beta_0) = \beta_0$.

We next discuss multifractal properties associated with the Hölder exponents. Consider the integral $u(x)$ of the invariant measure μ_0, defined by (3), which is plotted in Fig. 2. Since the attractor is topologically a Cantor set, the function $u(x)$ is a variant of the Devil's staircase (see ref. 13, Section 8.2).

To find its spectrum of Hölder exponents, notice that for every interval $\Delta_i^{(n)}$ the increase of $u(x)$ along the interval is equal to 2^{-n}. Hence $\Delta_i^{(n)}$ corresponds to a Hölder exponent $h_n(i) = \alpha$ where $|\Delta_i^{(n)}|^\alpha \sim \frac{1}{2^n}$. This implies

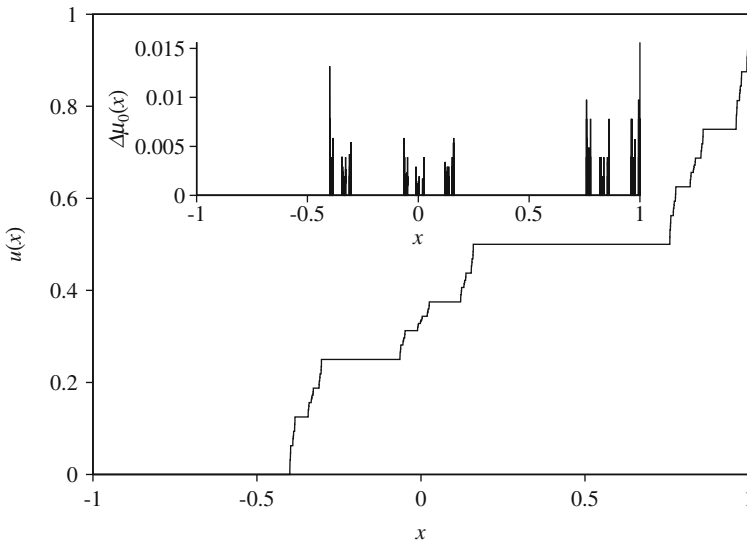$$|\Delta_i^{(n)}| \sim 2^{-\frac{n}{\alpha}} . \tag{28}$$

Fig. 2. The integral $u(x)$ of the Feigenbaum invariant measure calculated with 2^{15} bins of uniform length $\Delta = 2^{-14}$ in $[-1, 1]$. Inset: The invariant measure smoothed over the distance Δ calculated as a frequency histogram.

Using (21) and (28) we conclude that the Hölder exponent α corresponds to an inverse temperature $\beta(\alpha)$ such that

$$F'(\beta(\alpha)) = -\frac{\ln 2}{\alpha} . \tag{29}$$

This gives

$$\beta(\alpha) = (F')^{-1}\left(-\frac{\ln 2}{\alpha}\right) , \tag{30}$$

where $(F')^{-1}(y)$ is the inverse function to $F'(x)$. We can now find the Hausdorff dimension $d_H(\alpha)$ of the set of points $x \in A$ for which the Hölder exponent of $u(x)$ is equal to α:

$$d_H(\alpha) = d_H(\beta(\alpha)) = \beta(\alpha) - \frac{F(\beta(\alpha))}{F'(\beta(\alpha))} . \tag{31}$$

Notice that the analysis presented above can be made completely rigorous (see, for example, refs. 6 and 14).

2.2. Connection between Thermodynamic Formalism and Standard Multifractal Analysis

It is quite interesting to compare the multifractal analysis which we presented above with the one introduced in ref. 9. Although we restrict ourselves here to the case of the Feigenbaum attractor, the result presented below holds in a much more general setting. Basically our analysis is valid whenever the system under consideration can be described with the help of the thermodynamic formalism.

The basic object for our analysis is the Feigenbaum attractor itself and the method is based on the construction of the thermodynamic formalism for the lengths of the elements of dynamic partitions $\Delta_i^{(n)}$. The thermodynamic formalism uses considerable dynamical information about the map g. In contrast, the analysis in ref. 9 is carried out for fractal measures and does not directly use the dynamical information about the system. In the period-doubling setting the fractal measure is μ_0. It is the unique invariant measure for g acting on A (see ref. 6). It also can be considered as a physical or Sinai–Ruelle–Bowen (SRB) measure on $[-1, 1]$. This means that under dynamics given by the map g any initial absolutely continuous distribution $\bar{\mu}$ on $[-1, 1]$ converges to μ_0: $g^{(n)}(\bar{\mu}) \to \mu_0$ as $n \to \infty$. The multifractal analysis in ref. 9 is based on a function D_p which can

be defined in the following way. Consider a partition of the interval $[-1, 1]$ into subintervals $I_j,\ 1 \leqslant j \leqslant N$ of length l. Then

$$D_p = \lim_{l \to 0} \left(\frac{1}{p-1} \frac{\ln \sum_{j=1}^{N} (\mu_0(I_j))^p}{\ln l} \right) = \frac{\alpha}{\ln 2} F(\beta(\alpha)). \tag{32}$$

It follows from (32) that

$$\sum_{j=1}^{N} (\mu_0(I_j))^p \sim N^{D_p(1-p)}. \tag{33}$$

Another characteristic of a multifractal measure is given by its spectrum of dimensions $f(\alpha)$ which is just the Legendre transform of $D_p(p-1)$:

$$f(\alpha) = \inf_p\ [\alpha p - D_p(p-1)]. \tag{34}$$

The dual Legendre relation allows one to find $D_p(p-1)$ from $f(\alpha)$:

$$D_p(p-1) = \inf_\alpha\ [\alpha p - f(\alpha)]. \tag{35}$$

We next find a correspondence between the pair $(D_p(p-1),\ f(\alpha))$ and the pair of thermodynamic functions $(F(\beta),\ d_H(\alpha))$. We shall show that

$$D_p(p-1) = -F^{-1}(p \ln 2), \quad f(\alpha) = d_H(\alpha), \tag{36}$$

where F^{-1} is an inverse function to the free energy $F(\beta)$. To derive the first relation we consider the dynamical partition $\Delta_i^{(n)}, 0 \leqslant i \leqslant 2^n - 1$ and assume that $n \gg 1$ but $l \ll \min_i |\Delta_i^{(n)}|$. For each $\Delta_i^{(n)}$ define

$$M(\Delta_i^{(n)}) = \sum_{j: I_j \subset \Delta_i^{(n)}} (\mu_0(I_j))^p. \tag{37}$$

Notice that the asymptotic behavior of $M(\Delta_i^{(n)})$ depends only on asymptotic scalings of smaller elements of the dynamical partitions $\Delta_s^{(n+m)}$ inside $\Delta_i^{(n)}$. The thermodynamic formalism constructed above implies that asymptotically those scalings are completely determined by the potential U

and hence they do not depend on i. Rescaling the invariant measure inside $\Delta_i^{(n)}$ by a factor 2^n we conclude that

$$M(\Delta_i^{(n)}) = \frac{1}{2^{np}} \sum_{j:I_j \subset \Delta_i^{(n)}} (2^n \mu_0(I_j))^p \sim \frac{1}{2^{np}} N_i^{D_q(1-p)}, \tag{38}$$

where N_i is a total number of the intervals I_j inside $\Delta_i^{(n)}$. Taking the sum over i and using (14) we have

$$
\begin{aligned}
\sum_{j=1}^N (\mu_0(I_j))^p = \sum_{0 \leqslant i \leqslant 2^n - 1} M(\Delta_i^{(n)}) &\sim \frac{1}{2^{np}} \sum_{0 \leqslant i \leqslant 2^n - 1} N_i^{D_p(1-p)} \\
&\sim \frac{1}{2^{np}} \sum_{0 \leqslant i \leqslant 2^n - 1} \left(\frac{|\Delta_i^{(n)}|}{l} \right)^{D_p(1-p)} \\
&\sim \frac{1}{2^{np}} N^{D_p(1-p)} \exp[n F(D_p(1-p))].
\end{aligned}
\tag{39}
$$

This together with (33) immediately gives

$$\exp[F(D_p(1-p))] = 2^p \tag{40}$$

which implies the first relation in (36). We next show that the second relation holds. Using (34) we have

$$
\begin{aligned}
f(\alpha) = \inf_p \; [\alpha p - D_p(p-1)] = \inf_p \; [\alpha p - (-F^{-1}(p \ln 2)] &= \inf_z \; \left[\frac{\alpha}{\ln 2} z + F^{-1}(z) \right] \\
= \inf_\beta \; \left[\frac{\alpha}{\ln 2} F(\beta) + \beta \right] &= \frac{\alpha}{\ln 2} \inf_\beta \; \left[\frac{\ln 2}{\alpha} \beta + F(\beta) \right].
\end{aligned}
\tag{41}
$$

It is easy to see that the extremum in (41) corresponds to $\beta(\alpha) = (F')^{-1}\left(-\frac{\ln 2}{\alpha} \right)$ which implies

$$f(\alpha) = \beta(\alpha) + \frac{\alpha}{\ln 2} F(\beta(\alpha)) = d_H(\alpha). \tag{42}$$

Finally we express the scaling exponents ζ_p for the structure functions through the thermodynamic characteristics. The exponent ζ_p is defined by the scaling relation

$$\langle |u(x+l) - u(x)|^p \rangle \sim l^{\zeta_p} \tag{43}$$

in terms of the integral $u(x)$ of the invariant measure. Let I_j, $1 \leqslant j \leqslant N = 2/l$ be a partition of $[-1, 1]$ into intervals of length l. Then

$$\langle |u(x+l) - u(x)|^p \rangle = \frac{1}{N} \sum_{j=1}^{N} (\mu_0(I_j))^p \sim l N^{D_p(1-p)} \sim l^{1+D_p(p-1)} \quad (44)$$

which together with (36) gives

$$\zeta_p = 1 + D_p(p-1) = 1 - F^{-1}(p \ln 2). \quad (45)$$

Using (35) one can also write ζ_p in the following form:

$$\zeta_p = 1 + \inf_\alpha [\alpha p - f(\alpha)] = 1 + \inf_\alpha [\alpha p - d_H(\alpha)]. \quad (46)$$

At $\alpha = \alpha_\star$ the infimum (46) is attained, we can then write the relation between p and $\alpha_\star$ as

$$p = d'_H(\alpha)|_{\alpha=\alpha_\star} = \frac{F(\beta(\alpha_\star))}{\ln 2}. \quad (47)$$

The scaling exponents for the structure functions are hence obtained as

$$\zeta_p = 1 - \beta(\alpha_\star). \quad (48)$$

We now turn to concrete calculations for the scheme presented above. It is easy to see that all the thermodynamic functions can be effectively approximated numerically. The first step is to find approximations for the thermodynamic potential $U(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(n)}, \ldots)$. We shall use Markov approximations $U_k(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)})$ which are defined by the following formula:

$$U_k(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)}) = \lim_{n \to \infty} \ln \frac{\left| \Delta_{1,0,\ldots,0,\epsilon^{(k)},\epsilon^{(k-1)},\ldots,\epsilon^{(1)}}^{(n)} \right|}{\left| \Delta_{1,0,\ldots,0,\epsilon^{(k)},\epsilon^{(k-1)},\ldots,\epsilon^{(2)}}^{(n-1)} \right|}. \quad (49)$$

It was shown in ref. 6 that the limit in (49) exists and

$$|U(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)}, \epsilon^{(k+1)}, \ldots) - U_k(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)})| \leqslant C(2\gamma)^k. \quad (50)$$

Using the approximate potential U_k we can construct a Markov approximation for the transfer-matrix operator $\mathcal{L}(\beta)$. Namely, we define a finite dimensional linear operator $\mathcal{L}_k(\beta)$:

$$\mathcal{L}_k(\beta)h(\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)}) = \sum_{\epsilon^{(0)}=0,1} \exp[\beta U_{k+1}(\epsilon^{(0)}, \epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)})]$$
$$\times h(\epsilon^{(0)}, \epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k-1)}). \quad (51)$$

In the matrix representation the operator $\mathcal{L}_k(\beta)$ corresponds to a certain $2^k \times 2^k$ matrix. Denote by $\lambda_K(\beta)$ its largest eigenvalue. Then

$$F_k(\beta) = \ln \lambda_k(\beta) \quad (52)$$

is a natural approximation for the free energy $F(\beta)$. It follows from (50) that $F_k(\beta)$ converges to $F(\beta)$ exponentially fast in C^∞ topology. Using $F_k(\beta)$ we can effectively approximate all the multifractal functions which we discussed above. The corresponding numerical results are presented in the next section.

2.3. Numerical Calculation of the Free Energy and the Scaling Exponents

Here we show how to construct the transfer-matrix operator $\mathcal{L}_k$ starting from $k=0$ to general k. Note that transfer-matrix calculations for the cases $k=0, 1$ were also considered by M. Feigenbaum.[10,11] For $k=0$, the matrix operator $\mathcal{L}_0$ is just a scalar. Denoting $\Delta^{(n)}_{\epsilon_1, \ldots, \epsilon_n}$ by $\Delta^{(n)}(\epsilon_1, \ldots, \epsilon_n)$ for clarity, let us consider

$$\exp[U_1(\epsilon^{(0)})] = \lim_{n\to\infty} \frac{|\Delta^{(n)}(1, \overbrace{0, \ldots, 0}^{n-2}, \epsilon^{(0)})|}{|\Delta^{(n-1)}(1, \underbrace{0, \ldots, 0}_{n-2})|}, \quad (53)$$

whose analytical expression is easy to calculate. The 0-th order approximation of the free energy is given by

$$F_0(\beta) = \ln[e^{\beta U_1(0)} + e^{\beta U_1(1)}]. \quad (54)$$

For $\epsilon^{(0)} = 0$, since $\Delta^{(n)}_0 = [-\alpha^{-n}, \alpha^{-n}]$ and $\Delta^{(n)}_1 = [g(\alpha^{-n}), 1]$, we have

$$|\Delta^{(n)}(1, 0, \ldots, 0)| = 1 - g(\alpha^{-n}) = c\alpha^{-2n} + O(\alpha^{-4n}), \quad (55)$$

where we use the expansion (5) of $g(x)$. Thus $c = 1.527\ldots$. Therefore, the corresponding component of $\mathcal{L}_0(\beta = 1)$ is

$$e^{U_1(0)} = \lim_{n\to\infty} \frac{|c\alpha^{-2n} + O(\alpha^{-4n})|}{|c\alpha^{-2(n-1)} + O(\alpha^{-4(n-1)})|} = \alpha^{-2}. \tag{56}$$

Before considering the second term in the argument of the logarithm in (54), we recall the relation

$$g^{(2^n)}(\alpha^{-n}x) = (-1)^n\alpha^{-n}g(x), \tag{57}$$

which played a primary role in the proof of Theorem 4.1. of ref. 6. Since $\Delta^{(n)}(1, 0, \ldots, 0, 1) = \Delta^{(n)}_{1+2^{n-1}}$, in view of (57), we have

$$\begin{aligned}
|\Delta^{(n)}(1, 0, \ldots, 0, 1)| &= |g^{(1+2^{n-1})}(0) - g^{(1+2^{n-1})}(\alpha^{-n})| \\
&= |c\alpha^{-2(n-1)}\{[g(\alpha^{-1})]^2 - 1\} + O(\alpha^{-4(n-1)})|. \tag{58}
\end{aligned}$$

Thus for $\epsilon^{(0)} = 1$, (53) can be rewritten as

$$e^{U_1(1)} = \lim_{n\to\infty} \frac{|c\alpha^{-2(n-1)}\{[g(\alpha^{-1})]^2 - 1\} + O(\alpha^{-4(n-1)})|}{|c\alpha^{-2(n-1)} + O(\alpha^{-4(n-1)})|} = 1 - [g(\alpha^{-1})]^2. \tag{59}$$

We now arrive at the expression for the 0-th order approximation of the free energy:

$$F_0(\beta) = \ln[\alpha^{-2\beta} + \{1 - [g(\alpha^{-1})]^2\}^\beta]. \tag{60}$$

Next, consider the $k = 1$ approximation to the transfer matrix $\mathcal{L}_1(\beta)$. From (51), it can be written in standard matrix notation as

$$\begin{pmatrix} e^{\beta U_2(0,0)} & e^{\beta U_2(1,0)} \\ e^{\beta U_2(0,1)} & e^{\beta U_2(1,1)} \end{pmatrix}. \tag{61}$$

It then follows that the free energy is given by

$$\begin{aligned}
F_1(\beta) &= \ln\left[e^{\beta U_2(0,0)} + e^{\beta U_2(1,1)} + \sqrt{(e^{\beta U_2(0,0)} - e^{\beta U_2(1,1)})^2 + 4e^{\beta U_2(1,0)}e^{\beta U_2(0,1)}}\right] \\
&\quad - \ln 2. \tag{62}
\end{aligned}$$

The four components of the transfer matrix require the evaluation of suitable exponential terms, expressible by (49), from

$$\exp[U_2(\epsilon^{(0)}, \epsilon^{(1)})] = \lim_{n\to\infty} \frac{|\Delta^{(n)}(1, \overbrace{0, \ldots, 0}^{n-3}, \epsilon^{(1)}, \epsilon^{(0)})|}{|\Delta^{(n-1)}(1, \underbrace{0, \ldots, 0}_{n-3}, \epsilon^{(1)})|}. \tag{63}$$

Each of these terms is calculated in the same manner as for the $k=0$ case:

$$e^{U_2(0,0)} = \lim_{n\to\infty} \frac{|\Delta^{(n)}(1, 0, \ldots, 0)|}{|\Delta^{(n-1)}(1, 0, \ldots, 0)|} = \alpha^{-2}, \tag{64}$$

$$\begin{aligned} e^{U_2(1,0)} &= \lim_{n\to\infty} \frac{|\Delta^{(n)}(1, 0, \ldots, 0, 1)|}{|\Delta^{(n-1)}(1, 0, \ldots, 0)|} = [g(0)]^2 - [g(\alpha^{-1})]^2 \\ &= 1 - [g(\alpha^{-1})]^2, \end{aligned} \tag{65}$$

$$\begin{aligned} e^{U_2(0,1)} &= \lim_{n\to\infty} \frac{|\Delta^{(n)}(1, 0, \ldots, 0, 1, 0)|}{|\Delta^{(n-1)}(1, 0, \ldots, 0, 1)|} \\ &= \frac{|[g(0)]^2 - [g(\alpha^{-2})]^2|}{|[g(0)]^2 - [g(\alpha^{-1})]^2|} = \frac{1 - [g(\alpha^{-2})]^2}{1 - [g(\alpha^{-1})]^2}, \end{aligned} \tag{66}$$

$$\begin{aligned} e^{U_2(1,1)} &= \lim_{n\to\infty} \frac{|\Delta^{(n)}(1, 0, \ldots, 0, 1, 1)|}{|\Delta^{(n-1)}(1, 0, \ldots, 0, 1)|} \\ &= \frac{|[g^{(3)}(0)]^2 - [g^{(3)}(\alpha^{-2})]^2|}{|[g(0)]^2 - [g(\alpha^{-1})]^2|}. \end{aligned} \tag{67}$$

For $k=2$, we just write down the transfer matrix $\mathcal{L}_2(\beta)$:

$$\begin{pmatrix} e^{\beta U_3(0,0,0)} & 0 & e^{\beta U_3(1,0,0)} & 0 \\ e^{\beta U_3(0,0,1)} & 0 & e^{\beta U_3(1,0,1)} & 0 \\ 0 & e^{\beta U_3(0,1,0)} & 0 & e^{\beta U_3(1,1,0)} \\ 0 & e^{\beta U_3(0,1,1)} & 0 & e^{\beta U_3(1,1,1)} \end{pmatrix}. \tag{68}$$

Now we are in a position to calculate the transfer matrix $\mathcal{L}_k(\beta)$ for general k. Let us consider the component for $\beta = 1$:

$$\exp[U_{k+1}(\epsilon^{(0)}, \epsilon^{(1)}, \ldots, \epsilon^{(k)})] = \lim_{n\to\infty} \frac{|\Delta^{(n)}(1, \overbrace{0, \ldots, 0}^{n-k-2}, \epsilon^{(k)}, \ldots, \epsilon^{(1)}, \epsilon^{(0)})|}{|\Delta^{(n-1)}(1, \underbrace{0, \ldots, 0}_{n-k-2}, \epsilon^{(k)}, \ldots, \epsilon^{(1)})|}$$

$$= \lim_{n\to\infty} \frac{|g(\Delta^{(n)}(\overbrace{0,\dots,0}^{n-k-2}, \epsilon^{(k)}, \dots, \epsilon^{(1)}, \epsilon^{(0)}))|}{|g(\Delta^{(n-1)}(\underbrace{0,\dots,0}_{n-k-2}, \epsilon^{(k)}, \dots, \epsilon^{(1)}))|}. \tag{69}$$

Hence it is enough to calculate $|\Delta^{(n)}(0,\dots,0,\epsilon^{(k)},\dots,\epsilon^{(1)},\epsilon^{(0)})|$. For this we use the following relation:

$$|\Delta^{(n)}(\underbrace{0,\dots,0}_{n-k-2},\epsilon^{(k)},\dots,\epsilon^{(1)},\epsilon^{(0)})| = |g^{(\epsilon^{(k)}2^{n-k-1}+\dots+\epsilon^{(1)}2^{n-2}+\epsilon^{(0)}2^{n-1})}(0)$$
$$- g^{(\epsilon^{(k)}2^{n-k-1}+\dots+\epsilon^{(1)}2^{n-2}+\epsilon^{(0)}2^{n-1})}(\alpha^{-n})|$$
$$= \alpha^{-(n-k)}|\Delta^{(k+1)}(\epsilon^{(k)},\dots,\epsilon^{(1)},\epsilon^{(0)})|. \tag{70}$$

By (70) the numerator in (69) is given by

$$|\Delta^{(n)}_{1+j\cdot2^{n-k-1}}| = |\{1 - c\alpha^{-2(n-k-1)}[g^{(j)}(0)]^2\}$$
$$- \{1 - c\alpha^{-2(n-k-1)}[g^{(j)}(\alpha^{-(k+1)})]^2\} + O(\alpha^{-4(n-k)})|\}$$
$$= c\alpha^{-2(n-k-1)}|[g^{(j)}(0)]^2 - [g^{(j)}(\alpha^{-(k+1)})]^2| + O(\alpha^{-4(n-k)}). \tag{71}$$

Therefore the component (69) is expressed as

$$\exp[U_{k+1}(\epsilon^{(0)}, \epsilon^{(1)}, \dots, \epsilon^{(k)})] = \left| \frac{[g^{(j)}(0)]^2 - [g^{(j)}(\alpha^{-(k+1)})]^2}{[g^{(j')}(0)]^2 - [g^{(j')}(\alpha^{-k})]^2} \right|, \tag{72}$$

where $j = \epsilon^{(k)} + \epsilon^{(k-1)}2 + \dots + \epsilon^{(1)}2^{k-1} + \epsilon^{(0)}2^k$ and $j' = \epsilon^{(k)} + \epsilon^{(k-1)}2 + \dots + \epsilon^{(1)}2^{k-1}$. In other words, if the coordinates of the end points of the intervals are given by $\Delta^{(k+1)}_j = [a^{(k+1)}_j, b^{(k+1)}_j]$ and $\Delta^{(k)}_{j'} = [a^{(k)}_{j'}, b^{(k)}_{j'}]$, the component (69) can be rewritten as

$$\exp[U_{k+1}(\epsilon^{(0)}, \epsilon^{(1)}, \dots, \epsilon^{(k)})] = \left| \frac{\left(a^{(k+1)}_j\right)^2 - \left(b^{(k+1)}_j\right)^2}{\left(a^{(k)}_{j'}\right)^2 - \left(b^{(k)}_{j'}\right)^2} \right|. \tag{73}$$

If the transfer matrix is written as an ordinary $2^k \times 2^k$ matrix, it is easily found that the row and columns indices of the component (69) are given by (row, column) $= (p, q)$, where

$$\text{(row)} \quad p = 1 + \epsilon^{(k)} + \epsilon^{(k-1)}2 + \cdots + \epsilon^{(1)}2^{k-1}, \tag{74}$$

$$\text{(column)} \quad q = 1 + \epsilon^{(k-1)} + \epsilon^{(k-2)}2 + \cdots + \epsilon^{(0)}2^{k-1}. \tag{75}$$

For numerical calculation of the functions $F_k(\beta)$ and ζ_p associated to $u(x)$, we use the expansion (5) for $g(x)$, as given in ref. 12. We include terms up to x^{32}. Numerical calculation is done with standard double precision (15 significant digits). For obtaining the largest eigenvalue, we use the power method for matrices.[15] The multiplication of the matrix is stopped when the relative error of the most dominant eigenvalue becomes less than 10^{-13}, thus giving 10^{-13} absolute error on the $F(\beta)$ function. An alternative approach, also based on the thermodynamic formalism and yielding 10-digit accuracy, may be found in ref. 16. Approximate free energies $F_k(\beta)$, with k up to 16, calculated by the transfer-matrix method are shown in Fig. 3. We note that the $k = 0$ approximation (60) already gives a reasonable estimate. The discrepancy of the free energy between various orders of approximations is visible at large β. However the $\beta > 0$ region is irrelevant as far as ζ_p for positive p is concerned (see Eq. (48)). The corresponding ζ_p ($p \geqslant 0$) are calculated from the $F_k(\beta)$ by (45) for different values of k; the results, which hardly depend on k, are shown in Fig. 4(a). We also determined the structure functions of $u(x)$ with 2^{17}
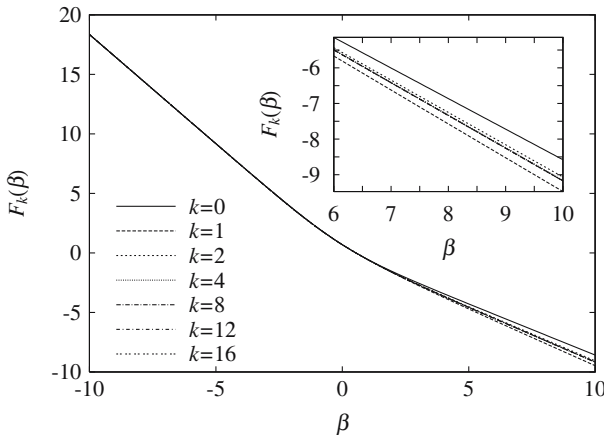

Fig. 3.   The k-th order approximation to the free energy $F_k(\beta)$. Inset: Enlargement of the region $6 \leq \beta \leq 10$; when increasing k, alternate convergence is observed.
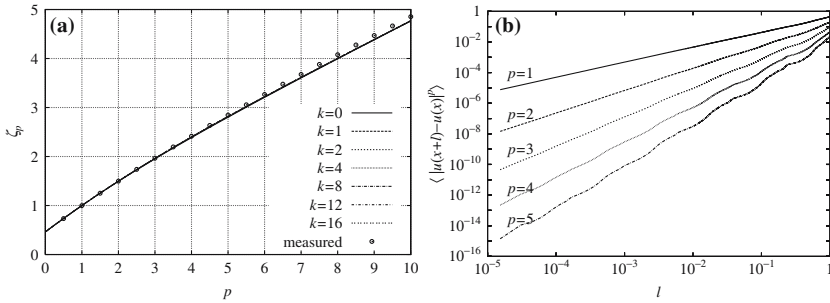
Fig. 4. (a) Scaling exponents ζ_p of structure functions obtained by two methods. Open circles: data obtained by a least-square fit of the slopes of the directly measured structure functions shown in (b). Lines: data calculated from the free energy $F_k(\beta)$ using (45) (curves for different k are essentially indistinguishable).

uniform bins in $[-1, 1]$; they are plotted in Fig. 4(b). The exponents ζ_p are then obtained by a least square fit of the structure functions over the range $2^{-16} \sim 1.5 \times 10^{-6} \leqslant r \leqslant 1$. With this number of bins, the quality of the fit begins to somewhat deteriorate beyond $p = 4$, but otherwise there is rather good agreement between the two methods of determining ζ_p. Note that the "y-intercept" of the graph of ζ_p, namely ζ_0, which is the codimension of the support of the invariant measure $\mu_0(dx)$, is positive and its numerical value is slightly under one half.[6] This will be important in the sequel.

3. FRACTIONAL DERIVATIVES FOR THE FEIGENBAUM ATTRACTOR

3.1. Phenomenology for Multifractality and Fractional Derivatives

In this section we briefly recall the phenomenological approach to multifractality via fractional derivatives[1] and adapt it to a multifractal measure. We therefore work, not with the measure $\mu_0(dx)$ itself, but with its integral $u(x)$. Singularity exponents α may be viewed as local Hölder exponents of $u(x)$, i.e., $|u(x+l) - u(x)| \propto |l|^\alpha$ for $(l \to 0)$. We turn to fractional derivatives of order a defined, as in ref. 1, as the multiplication in the Fourier space by $\hat{u}(k)$ by $|k|^a$ (see ref. 1 for precise definition). An isolated non-oscillatory singularity with exponent α at a point x implies

$$|D^a u(y)| \sim |y - x|^{\alpha - a} \quad (y \to x). \tag{76}$$

If $\alpha - a < 0$, as we shall assume hereafter, the exponent is negative, the fractional derivative can become arbitrarily large and thus contributes to

the tail-behavior of the probability. A key assumption in the phenomenology is that this argument can be carried over to non-isolated multifractal singularities, provided we take all types of singularities into account. For the Feigenbaum invariant measure, we know the Hausdorff dimension $d_H(\alpha)$ of the set of points having a singularity with exponent α. Assuming that we can also use $d_H(\alpha)$ as a covering dimension, we can express the probability to have a singularity of exponent α contributing a fractional derivative of order a which exceeds (in absolute value) a given large value ξ, that is we require

$$|y - x| < \xi^{-\frac{1}{a-\alpha}}. \tag{77}$$

In terms of the codimension of the set $\mathcal{I}_\alpha$, the probability to satisfy (77) is written as

$$\text{Prob}\{|D^a u| > \xi\} \propto |y - x|^{d - d_H(\alpha)} \propto \xi^{-\frac{d - d_H(\alpha)}{a - \alpha}}. \tag{78}$$

Here d is the spatial dimension ($d = 1$). Taking now into account the singularities with all possible exponents α, the tail of the cumulative probability of the fractional derivative of order a is given, to the leading order, by the following power law

$$\text{Prob}\{|D^a u| > \xi\} \propto \xi^{-p_\star}, \quad \xi \to \infty, \tag{79}$$

$$p_\star = \inf_{\alpha < a} \frac{d - d_H(\alpha)}{a - \alpha}. \tag{80}$$

An easy calculation shows that $\alpha_\star$ corresponding to the infimum in (80) satisfies

$$\alpha_\star = \alpha_\star(a) = a + \frac{d_H(\alpha_\star) - d}{d'_H(\alpha_\star)}, \tag{81}$$

which immediately gives $p_\star = d'_H(\alpha_\star)$. On the other hand, we know that

$$\zeta_p = \inf_\alpha (p\alpha + d - d_h(\alpha)). \tag{82}$$

Here the infimum is given by an α satisfying the very same relation $p_\star = d'_H(\alpha_\star)$. Hence,

$$\zeta_{p_\star} = p_\star \alpha_\star + d - d_H(\alpha_\star). \tag{83}$$

Using (81), we get

$$\zeta_{p_\star} = p_\star \left(a + \frac{d_H(\alpha_\star) - d}{d'_H(\alpha_\star)} \right) + d - d_H(\alpha_\star) = p_\star a, \tag{84}$$

where the second relation follows from $p_\star = d'_H(\alpha_\star)$. The geometrical interpretation of this equation is that the (negative) exponent of the power-law tail for the fractional derivative of order a is the p-value of the intersection of the graph of ζ_p and of a straight line of slope a through the origin.

As shown in ref. 1, in the presence of the finite range of scaling, the power-law tail (79) emerges only if the multifractality is sufficiently strong. This strength is given by the multifractality parameter $C(a)$, a measure of how strongly the data depart from being self similar (which would imply $\zeta_p \propto p$):

$$C(a) \equiv a - \alpha_\star = \frac{\zeta_{p_\star}}{p_\star} - \frac{d\zeta_p}{dp}\bigg|_{p=p_\star}, \tag{85}$$

where $\alpha_\star \equiv d\zeta_p/dp|_{p=p_\star}$. It was shown that observability of the power-law requires a sufficiently large value for the product $nC(a)$, where n is the number of octaves over which the data present multifractal scaling. In practice it was found in ref. 1 that

$$nC(a) \geqslant 10. \tag{86}$$

For example, fully-developed turbulence velocity data[17] have typical C values of the order of 1/30, thereby requiring a monstrous inertial range of about 300 octaves for observability of power-law tails. As we shall see, the situation is much more favorable for the Feigenbaum invariant measure.

Before turning to numerical issues, we comment on an issue raised by an anonymous referee who worried about the nonlocal character of the fractional derivative and wrote in essence that our approach makes sense, strictly speaking, only for (statistically) translationally invariant in space systems: otherwise, if the system consists of components whose "fractal properties" are rather different the results will be smeared out. Our feeling about such matters is summarized as follows. First one can observe that, of course, the attractor for the Feigenbaum map is not homogeneous (translation-invariant) but after zooming in it becomes increasingly so; the fractional derivative is not a local operator but the tail of its PDF is likely to be dominated by strongly localized events. Second, a more technical

observation. The idea of the multi-fractal analysis is based on the fact that the dynamics of a system determines a variety of scales. It is important that these scales do not depend on a particular place in the phase space. On the contrary, they are present and "interact" with each other everywhere. In the case of the Feigenbaum attractor the scales depend on a symbolic location in a system of partitions. In physical systems, like homogeneous turbulence, such partitions are difficult to define rigorously. However, the invariance with respect to the space coordinate is still present and forms a basis for applicability of the multifractal calculus.

3.2. Numerical Analysis of Fractional Derivatives

The phenomenological arguments presented in the previous section suggest that we should find power-law tails in the cumulative probability for fractional derivatives of $u(x)$ for suitable orders a. Inspection of Fig. 4 indicates that a should be between the minimum slope of the graph and unity. The value $a = 1$ is of course not a fractional order but, as we shall see, it is associated with a power-law tail of exponent minus one.[6] The minimum slope can be easily found. Indeed, $F(\beta)$ takes large values when β is large negative. In this case the main contribution to $F(\beta)$ comes from the shortest interval of the partition with the length of the order of γ^{2n}. Hence, $F(\beta)/|\beta| \to -2\ln\gamma$ in the limit $\beta \to -\infty$. This gives the following lower bound of the differentiation order:

$$\lim_{p\to\infty} \frac{\zeta_p}{p} = \lim_{p\to\infty} \frac{1 - F^{-1}(p\ln 2)}{p} = -\frac{\ln 2}{2\ln\gamma} \approx 0.3777. \qquad (87)$$

Table I. Feigenbaum Invariant Measure: Scaling Exponents ζ_p, Inverse Temperature β, Multifractality Parameter C and Number of Scaling Octaves Needed

p	ζ_p	β	C	Number of octaves needed
1.0	1.0	0.0	0.48	21
1.5	1.2540292658	−2.54029265895	0.34	29
2.0	1.4985620106	−4.98562010659	0.28	36
2.5	1.7344372955	−7.34437295506	0.24	42
3.0	1.9625763533	−9.62576353333	0.22	45

[6]This is a consequence of $\zeta_1 = 1 - F^{-1}(\ln 2) = 1$. (Putting $\beta = 0$ into the partition function $Z_k(\beta) \sim \sum_{j=0}^{2^k-1} |\Delta_j^{(k)}|^\beta$, we have $F_k(0) = \ln Z_k(0)/k = \ln 2$.)

We have already observed that, because the ζ_p graph does not pass through the origin, substantial values can be expected for the multifractality parameter $C(a)$. The actual values of C, associated to values of p ranging from 1 to 3 by increments of 0.5 are shown in Table I, together with the number n of scaling octaves needed determined by $nC \approx 10$ (cf. Eq. (86)). In practice, on a 32 bit machine, we are limited to about
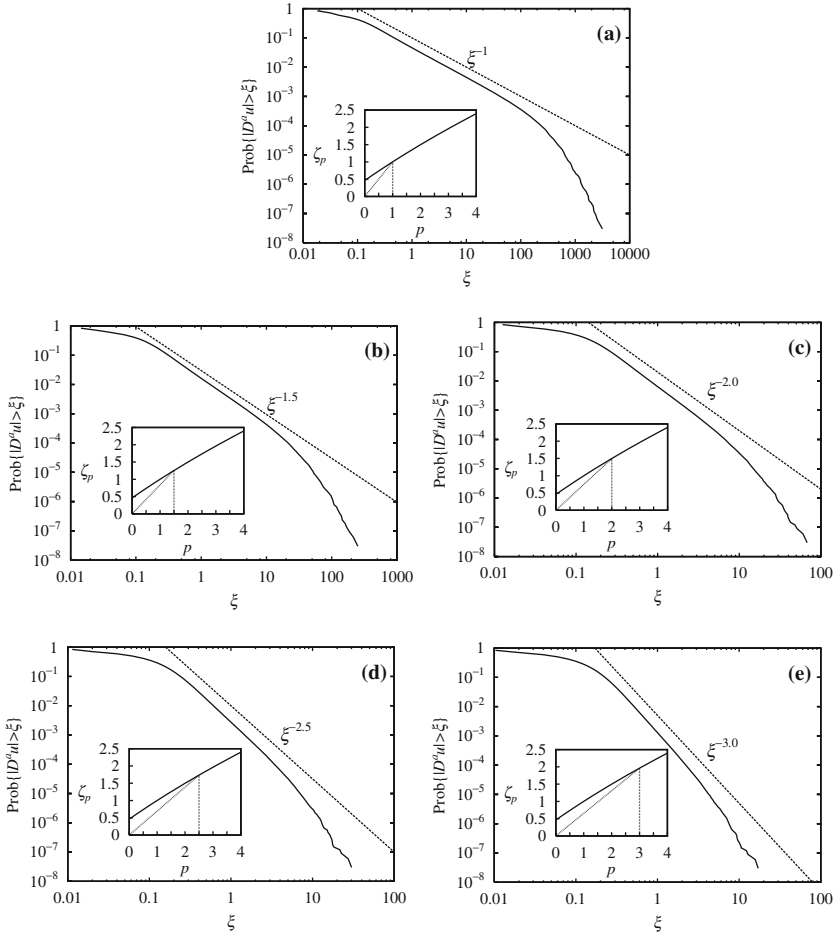

Fig. 5. Cumulative probabilities of absolute values of fractional derivatives of various orders $a = \zeta_{p_\star}/p_\star$ for the Feigenbaum invariant measure. Each function displays a power-law tail with an exponent fairly close to the predicted value $-p_\star$. Insets: corresponding ζ_p graphs. (a) $a = 1.0$ ($p_\star = 1.0$). (b) $a = 0.83$ ($p_\star = 1.5$). (c) $a = 0.74$ ($p_\star = 2.0$). (d) $a = 0.69$ ($p_\star = 2.5$). (e) $a = 0.65$ ($p_\star = 3.0$).

25 octaves of dynamical range in resolution over the interval $[-1, 1]$. This should be enough to observe power-law tails.

Indeed, Fig. 5 shows five instances of cumulative probabilities of fractional derivatives with power-law tails, corresponding to the values of the exponent p listed in Table I. The corresponding order of differentiation a ranges between 0.65 and 1.[7] Since the function $u(x)$ which we are analyzing is not periodic, we resort to the Hann windowing technique employed previously in ref. 1 (Section 13.4). Also, we use rank ordering to avoid binning.

The power-law behavior observed is consistent with the phenomenological theory presented in Section 3.1, the residual discrepancies being due to the resolution of 2^{25} bins.

4. CONCLUDING REMARKS

We have found solid numerical evidence for the presence of power-law tails in the cumulative distribution of fractional derivatives for the integral $u(x)$ of the invariant measure of the Feigenbaum map. Furthermore the exponents measured are consistent with those predicted by phenomenological arguments from the spectrum of singularities. Since we have a fairly deep understanding of the structure of the attractor, thanks in particular to the thermodynamic formalism, a reasonable goal may be to actually prove the results. The main difficulty is that the operation of fractional derivative is non-local. However, we believe that a rigorous analysis here is still possible due to the quite simple spectral structure of the dynamical system corresponding to the Feigenbaum attractor.

ACKNOWLEDGMENTS

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[7]As predicted by the theory, when a is too small, e.g. for $a = 0$, no power-law tail is observed.

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